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A BOUNDARY LAYER THEORY FOR THE END PROBLEM OF A CIRCULAR CYLINDER

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SUMMARY

This paper discusses the nature of an approximate solution for the hollow circular cylinder whose fixed ends are given a uniform relative axial displacement and whose cylindrical surfaces are free from traction. We shall take the solution of this problem to be given by a super-position of the following two problems: problem I considers a finite length cylinder whose ends are given a relative axial displacement, but are no longer fixed; problem II removes the radial displacement at the end of the cylinder obtained in problem I.

1. *Introduction.*

Due to the great difficulty in obtaining exact solutions of boundary value problems according to the theory of three-dimensional elasticity, there exist in the literature various approximate theories governing the deformation of solids. The differential equations of these theories are an approximation to the three-dimensional equations and are obtained with the help of certain simplifying assumptions. For bodies of revolution whose thickness is very small compared to the other dimensions, the approximate theories are called shell theories. Here, by making *a priori* assumptions regarding the distribution of stresses and displacements with the thickness of the shell, we are able to obtain two-dimensional equations which describe the deformation in terms of quantities defined on the midsurface of the shell. But due to the fact that we are now dealing with middle surface force resultants and middle surface moments, we are able to satisfy the given boundary conditions only in a statically equivalent sense. Thus, for most problems, the approximate shell solution will differ markedly from the exact solution close to the boundary.

This paper discusses the nature of an approximate solution for the hollow circular cylinder whose fixed ends are given a uniform relative axial displacement and whose cylindrical surfaces are free from traction. We shall take the solution of this problem to be given by a super-position of the following two problems: problem I considers a finite length cylinder whose ends are given a relative axial displacement, but are no longer fixed; problem II removes the radial displacement at the end of the cylinder obtained in problem I.

The solution to problem I is straight-forward. The approximate solution of problem II is obtained by the application of the method of asymptotic integration to the three-dimensional equations of orthotropic elasticity. This method combines the application of the boundary layer technique with an expansion of the stresses and displacements in terms of a small geometric parameter. No prior assumptions regarding the thickness variation of the stresses and displacements are made. As a result we obtain two different sets of systems of differential equations. The lowest order systems correspond to the equations which govern the plane strain orthotropic elasticity theory and the isotropic shallow shell bending theory and constitute a "first approximation" to the solution of problem II. The plane strain theory gives a description close to the end of the shell and is termed the "boundary layer" solution, while the shell bending theory gives the "interior" solution. The higher order systems yield the higher order terms in the expansion and constitute thickness corrections. In the following, we will concern ourselves only with the solution of the first approximation

2. *Formulation.*

The problem of a transversely isotropic (orthotropic) cylinder subjected to equal and opposite end loads P (see Fig. 1) can be described by the

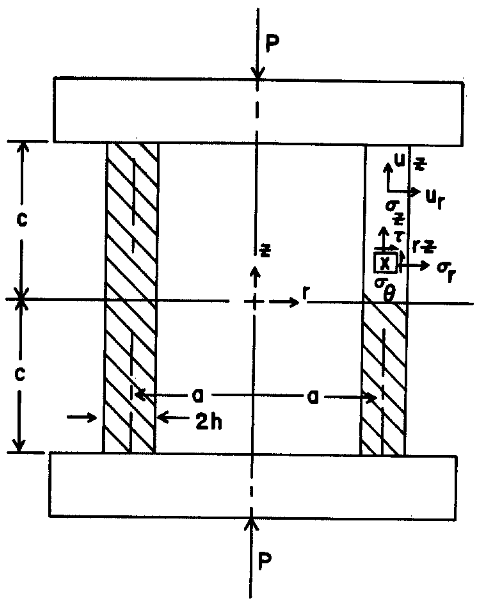

Fig 1. Circular cylinder under compressive load P

following equations:
EQUILIBRIUM EQUATIONS*

$$\frac{\partial}{\partial r}(r\sigma_r) + \frac{\partial}{\partial z}(r\tau_{rz}) - \sigma_\theta = 0$$
$$\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial}{\partial z}(r\sigma_z) = 0 \tag{1}$$

STRAIN-DISPLACEMENT EQUATIONS

$$\epsilon_r = \frac{\partial u_r}{\partial r}, \quad \epsilon_z = \frac{\partial u_z}{\partial z}, \quad \epsilon_0 = \frac{u_r}{r}$$
$$\gamma_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \tag{2}$$

CONSTITUTIVE EQUATIONS

*For Nomenclature, see p. 353.

$$\begin{aligned}
\epsilon_r &= \frac{\sigma_r}{E_t} - \frac{\nu_t}{E_t}(\sigma_z + \sigma_\theta) \\
\epsilon_z &= \frac{1}{E}(\sigma_z - \nu\sigma_\theta) - \frac{\nu_t}{E_t}\sigma_r \\
\epsilon_\theta &= \frac{1}{E}(\sigma_\theta - \nu\sigma_z) - \frac{\nu_t}{E_t}\sigma_r \\
\gamma_{rz} &= \frac{1}{G_t}\tau_{rz}
\end{aligned} \tag{3}$$

It is noted that the direction normal to the middle surface is the axis of elastic symmetry.

For boundary conditions we will assume that the cylindrical surfaces are free from traction,

$$\sigma_r = \tau_{rz} = 0 \qquad (r = a \pm h) \tag{4}$$

and that at the end we have

$$u_z = \mp K, \ u_r = 0 \qquad (z = \pm c) \tag{5}$$

Here, K is a constant which is chosen so that

$$\int_{a-h}^{a+h} \sigma_z(c, r) 2\pi r \, dr = - P \tag{6}$$

Conditions (4) and (5) approximate those usually found in an ordinary testing machine.

The solution to the problem thus formulated will now be obtained by a superposition of the solutions of the following two problems. The first problem consists of a cylinder with surface conditions (4) and on whose end we specify

$$u_z = \mp K, \ \tau_{rz} = 0 \ (z = \pm c) \tag{7}$$

where K is determined by (6). The solution of this problem is elementary and is given by

$$\begin{aligned}
\sigma_r^I &= \sigma_\theta^I = \tau_{rz}^I = 0, \ \sigma_z^I = -K\frac{E}{c} \\
u_z^I &= -K\frac{z}{c}, \ u_r^I = K\nu\frac{r}{c}
\end{aligned} \tag{8}$$

Here, the superscript I refers to problem one and

$$K = \frac{Pc}{4\pi a h E}. \tag{9}$$

For the second problem it will be convenient for us to shift the origin of the coordinate system to $z = -c$ and to define a new dimensionless variable by

$$z' = \frac{z + c}{L}, \tag{10}$$

where L is a small but as yet undetermined length scale in the axial direction. The second problem in terms of the variable z' now consists of a semi-infinite cylinder for whose cylindrical surfaces condition (4) holds and on whose end z' = 0 the following conditions hold:

$$u_z^{II} = 0, \quad u_r^{II} = - u_r^I \qquad (z' = 0). \tag{11}$$

We also prescribe that all stress components approach zero as z' becomes large,

$$\text{Stresses} \to 0 \text{ as } z' \to \infty. \tag{12}$$

From condition (12) it follows that the stress component σ_z is self-equilibriating at z' = 0,

$$\int_{a-h}^{a+h} \sigma_z^{II}(o, r)\, 2\pi r\, dr = 0. \tag{13}$$

An approximate solution to problem II will now be obtained by the method of asymptotic integration of equations (1), (2) and (3). One obtains as a result of the application of this method the correct approximate systems of equations needed to solve problem II. Although the method has been previously used by Reiss [1], Johnson and Reissner [2] and Widera [3] for other cylinder problems, it is presented again in the analysis to follow in order to make the paper self-contained.

3. *Asymptotic integration of the elasticity equations*

We introduce additional dimensionless coordinates, stresses and displacements as follows:

$$\rho = \frac{r-a}{h}$$

$$u_r = \frac{a}{E}(1-\nu^2)\sigma v_r, \quad u_z = \frac{a}{E}(1-\nu^2)\sigma v_z \tag{14}$$

$$\sigma_r = \sigma s_r, \quad \sigma_Z = \sigma s_z, \quad \sigma_\theta = \sigma s_\theta, \quad \tau_{rZ} = \sigma s_{rz}$$

where σ is a quantity having dimension of stress. In terms of these variables equations (1), (2) and (3) can be written in the following manner:

$$\begin{aligned}
(1-\nu^2)\dot{v}_r &= \lambda\left[\frac{\nu_n}{\nu_t} s_r - \nu_n(s_z + s_\theta)\right] \\
(1-\nu^2)\dot{v}_z &= -\frac{\lambda}{\mu}(1-\nu^2)v_r^I + 2\lambda(1+\nu)I_n s_{rz} \\
\frac{\lambda}{\mu}(1-\nu^2)v_z^1 &= \lambda(s_z - \nu s_\theta - \nu_n s_r) \\
(1-\nu^2)\frac{v_r}{1+\lambda\rho} &= s_\theta - \nu s_z - \nu_n s_r \\
\left[(1+\lambda\rho)s_{rz}\right]^{\cdot} &= -\frac{\lambda}{\mu}\left[(1+\lambda\rho)s_z\right]^I \\
\left[(1+\lambda\rho)s_r\right]^{\cdot} &= -\frac{\lambda}{\mu}\left[(1+\lambda\rho)s_{rz}\right]^I + \lambda s_\theta
\end{aligned} \tag{15}$$

Here,

$$\lambda = \frac{h}{a}, \quad \mu = \frac{L}{a}$$
$$\nu_n = \nu_t \frac{E_t}{E}, \quad I_n = \frac{E}{2(1+\nu)G_t}, \tag{16}$$

and $(\quad)^{\cdot}$ denotes differentiation with respect to ρ and $(\quad)^{I}$ differentiation with respect to z'.

Next we assume that for sufficiently small λ, i.e., $\lambda \ll 1$, we can expand each stress and displacement component asymptotically in a power series in $\lambda^{\frac{1}{2}}$. We must also choose L, or rather μ, as a function of λ. There are two possible choices [1, 3]:

$$(1)\ \mu = \lambda^{\frac{1}{2}}$$
$$(2)\ \mu = \lambda. \tag{17}$$

For $\mu = \lambda^{\frac{1}{2}}$, (10) becomes

$$z' = \frac{z+c}{(ha)^{\frac{1}{2}}} = \xi \tag{18}$$

and the power series expansion for the stresses and displacements is given by

$$s(z', \rho; \lambda) \sim \sum_{i=0}^{\infty} s^{(i)}(\xi, \rho)\left[\lambda^{\frac{1}{2}}\right]^{i}$$
$$v(z', \rho; \lambda) \sim \sum_{i=0}^{\infty} v^{(i)}(\xi, \rho)\left[\lambda^{\frac{1}{2}}\right]^{i} \tag{19}$$

For the case of $\mu = \lambda$, we set

$$z' = \frac{z+c}{h} = \eta \tag{20}$$

and use the following expansions:

$$s(z', \rho; \lambda) \sim \sum_{i=0}^{\infty} t^{(i)}(\eta, \rho)\left[\lambda^{\frac{1}{2}}\right]^{i}$$
$$v(z', \rho; \ ) \sim \sum_{i=0}^{\infty} \omega^{(i)}(\eta, \rho)\left[\lambda^{\frac{1}{2}}\right]^{i} \tag{21}$$

If $\mu = \lambda^{\frac{1}{2}}$ and expansions (19) are substituted into (15) we obtain, upon requiring that the equations be integrable with respect to ρ in a step-by-step manner, a series of systems of differential equations. The integrated form of the lowest order in $\lambda^{\frac{1}{2}}$ system (from now on called the first approximation) is given by

$$v_r^{(o)} = V_r^{(o)}(\xi), \quad v_z^{(1)} = V_z^{(1)}(\xi) - V_r^{(o)I}\rho$$
$$s_\theta^{(o)} = V_r^{(o)} + \nu\left[V_z^{(1)I} - V_r^{(o)II}\rho\right] \tag{22}$$
$$s_z^{(o)} = \nu V_r^{(o)} + V_z^{(1)I} - V_r^{(o)II}\rho$$

$$s_{rz}^{(1)} = S_{rz}^{(1)}(\xi) - \left[\nu V_r^{(o)I} + V_z^{(1)II}\right]\rho + V_r^{(o)III}\frac{\rho^2}{2}$$

$$s_r^{(2)} = S_r^{(2)}(\xi) + \left[-S_{rz}^{(1)I} + \nu V_z^{(1)I} + V_r^{(o)}\right]\rho + \qquad (22)\ \text{cont'd}$$

$$+ V_z^{(1)III}\frac{\rho^2}{2} - V^{(o)IV}\frac{\rho^3}{6}$$

where $V_r^{(o)}(\xi)$ and $V_z^{(1)}(\xi)$ are the middle surface displacements and $S_r^{(1)}(\xi)$ and $S_{rz}^{(2)}(\xi)$ are the middle surface stresses. The relative orders of magnitude are indicated by the superscript.

On substituting the expansions (21) into (15) and setting $\mu = \lambda$, we again obtain a series of systems of differential equations. These equations, though, are not integrable with respect to ρ in a step-by-step manner. The approximation equations can be written as

$$t_{rz}^{(o)} = \left(\frac{1-\nu}{2I_n}\right)\left[\omega_Z^{(2)\bullet} + \omega_r^{(2)I}\right]$$

$$t_r^{(o)} = K_{nt}\left[(1-\nu)\omega_r^{(2)\bullet} + \nu_n\omega_z^{(2)I}\right]$$

$$t_z^{(o)} = M_{nt}\,\omega_z^{(2)I} + \nu_n K_{nt}\,\omega_r^{(2)\bullet}$$

$$t_\theta^{(o)} = N_{nt}\,\omega_Z^{(2)I} + \nu_n K_{nt}\,\omega_r^{(2)\bullet} \qquad (23)$$

$$M_{nt}\,\omega_z^{(2)II} + \left(\frac{1-\nu}{2I_n}\right)\omega_z^{(2)\bullet\bullet} + \left[K_{nt}\nu_n + \left(\frac{1-\nu}{2I_n}\right)\right]\omega_r^{(2)I\bullet} = 0$$

$$\left(\frac{1-\nu}{2I_n}\right)\omega_r^{(2)II} + K_{nt}(1-\nu)\omega_r^{(2)\bullet\bullet} + \left[K_{nt}\nu_n + \left(\frac{1-\nu}{2I_n}\right)\right]\omega_z^{(2)I\bullet} = 0$$

Here,

$$K_{nt} = \frac{\frac{\nu_t}{\nu_n}(1-\nu^2)}{1-\nu-2\nu_n\nu_t} \qquad (24)$$

$$M_{nt} = 1 + \frac{\nu_n^2 K_{nt}}{1-\nu}, \quad N_{nt} = \nu + \frac{\nu_n^2 K_{nt}}{1-\nu}.$$

It should be pointed out that in order to obtain (23) it was necessary for the stresses to be of the same order and for the displacement to be of order λ relative to the stresses. A comparison of the first approximation equations (22) and (23) shows that the effect of transverse isotropy is present only in the equations associated with length scale h.

The boundary conditions for the functions $s^{(i)}(\xi,\rho)$, $v^{(i)}(\xi,\rho)$, $t^{(i)}(\eta,\rho)$ and $\omega^{(i)}(\eta,\rho)$ are obtained by substituting (19) and (21) into equations (4), (11) and (12). In particular, if the expansions for V_r and V_z are substituted into (11), we obtain for the first two nonzero expansion coefficients

$$v_z^{(1)}(o,\rho) = V_z^{(1)}(o) - V_r^{(o)}(o)\rho = 0$$

$$\omega_z^{(2)}(o,\rho) = 0 \qquad (25)$$

and

$$v_r^{(o)}\ (o,\rho) = V_r^{(o)}(o) = \frac{-\nu}{1-\nu^2}$$

$$v_r^{(2)}(o,\rho) + \omega_r^{(2)}(o,\rho) = \left\{V_r^{(2)}(o) + \frac{\nu_n}{1-\nu}\left[-V_r^{(o)}(o)\rho - \right.\right.$$

$$\left.\left. - V_z^{(1)I}(o)\rho + V^{(o)II}\frac{\rho^2}{2}\right]\right\} + \omega_r^{(2)}(o,\rho) = -\frac{\nu}{1-\nu^2}\rho \tag{26}$$

where the relation

$$v_r^{(2)}(\xi,\rho) = V_r^{(2)}(\xi) + \frac{\nu_n}{1-\nu}\left\{-\left[V_r^{(o)} + V_z^{(1)I}\right]\rho + V_r^{(o)II}\frac{\rho^2}{2}\right\} \tag{27}$$

has been used. This relation is obtained from the second approximation equations for $\mu = 1$. Also, in order that dimensionless quantities be 0(1) we have chosen

$$\sigma = \frac{P}{4\pi a h}. \tag{28}$$

It follows from (25) and (26) that the conditions for $V_r^{(o)}(\xi)$, $\omega_r^{(2)}(\eta,\rho)$ and $\omega_z^{(2)}(\eta,\rho)$ at $\xi = \eta = 0$ are

$$\begin{aligned} V_r^{(o)} &= -\frac{\nu}{1-\nu^2} \\ V_r^{(o)I} &= 0 \end{aligned} \tag{29}$$

and

$$\omega_z^{(2)}(o,\rho) = 0$$

$$\omega_r^{(2)}(o,\rho) = -\frac{\nu}{1-\nu^2}\rho - \frac{\nu_n}{1-\nu}\left\{-\left[V_r^{(o)}(o) + V_z^{(1)}(o)\right]\rho + \right.$$

$$\left. + V_r^{(o)III}(o)\frac{\rho^2}{2}\right\} \tag{30}$$

where we have set

$$V_r^{(2)}(\xi) = 0$$

in order to admit only decay type solutions.

Boundary condition (4) is to be satisfied by each term of the respective stress expansions.

We will now consider the solution of (22) and (23) subject to the boundary conditions stated above. The solution of (22) will constitute the "interior" solution of problem II, while that of (23) will represent the "boundary layer" solution.

4. *Solution of the interior problem*

Application of (4) to equations (22) yields

$$S_{rz}^{(1)} = -\tfrac{1}{2}V_r^{(o)III} \tag{31}$$

$$\nu V_r^{(o)I} + V_z^{(1)II} = 0$$

$$S_r^{(2)} = -\tfrac{1}{2}\, V_z^{(1)III}$$
$$V_r^{(o)IV} + 4\,\gamma^4 V_r^{(o)} = 0, \tag{31 cont'd}$$

where

$$\gamma^4 = \frac{3(1-\nu^2)}{4}. \tag{32}$$

The form of the solution of (31), 4 subject to condition (12) is given by

$$V_r^{(o)} = e^{-\gamma\xi}(A_1 \cos\gamma\xi + A_2 \sin\gamma\xi). \tag{33}$$

Satisfaction of boundary conditions (29) by (33) yields

$$A_1 = A_2 = -\frac{\nu}{1-\nu^2}. \tag{34}$$

The final expression for $V_r^{(o)}(\xi)$ is thus

$$V_r^{(o)} = -\frac{\nu}{1-\nu^2}(\cos\gamma\xi + \sin\gamma\xi)e^{-\gamma\xi}. \tag{35}$$

5. *Solution of the boundary layer problem*

We now consider the solution of equations (23) subject to boundary conditions (30) and

$$t_r^{(o)}(\eta, \pm 1) = t_{rz}^{(o)}(\eta, \pm 1) = 0 \tag{36}$$

$$\text{Stresses} \to 0 \text{ as } \eta \to \infty. \tag{37}$$

The set of equations (23), (30), (36) and (37) corresponds to the problem of an orthotropic semi-infinite strip undergoing a given end displacement. For the case of an isotropic material, the problem of the semi-infinite strip has previously been studied by Horvay [4] and Johnson and Little [5]. A class of end problems for the dynamic case has been investigated by Wu and Plunkett [6]. The method used in the latter investigation will be used in the following to formulate the solution of the posed problem.

Let us seek a solution of (23) in the form

$$\omega_z^{(2)}(\eta, \rho) = a_o + \sum_{n=1}^{\infty} a_n u_1^n(\rho;\beta_n)e^{-\beta_n\eta}$$
$$\omega_r^{(2)}(\eta, \rho) = \sum_{n=1}^{\infty} a_n u_2^n(\rho, \beta_n)e^{-\beta_n\eta}, \tag{38}$$

where the β_n's (Re $\beta_n > 0$) are the roots of a characteristic equation to be derived from the homogeneous boundary conditions (36). The constant a_o and the complex constants a_n must be determined so as to satisfy the displacement boundary conditions (30). On substituting (38) into (23) we obtain for the symmetric solution

$$u_1^n(\rho;\beta_n) = C(\beta_n)\cosh\alpha_1(\beta_n)\rho + \cosh\alpha_2(\beta_n)$$
$$u_2^n(\rho;\beta_n) = B_1(\beta_n)\,C(\beta_n)\sinh\alpha_1(\beta_n)\rho +$$
$$+ B_2(\beta_n)\sinh\alpha_2(\beta_n)\rho, \tag{39}$$

and for the antisymmetric solution

$$
\begin{aligned}
u_1^n(\rho,\beta_n) &= C(\beta_n)\ \sinh \alpha_1(\beta_n)\rho\ +\ \sinh \alpha_2(\beta_n)\rho \\
u_2^n(\rho;\beta_n) &= B_1(\beta_n)\ C(\beta_n)\ \cosh \alpha_1(\beta_n)\rho\ + \\
&\quad +\ B_2(\beta_n)\ \cosh \alpha_2(\beta_n)\rho,
\end{aligned}
\tag{40}
$$

Here, α_1^2 and α_2^2 are the roots of the equation

$$
\alpha^4 + \frac{2(I_n-\nu)}{1-\nu}\ \beta_n^2\alpha^2 + \frac{\nu(1-\nu\nu_t)}{\nu_t(1-\nu^2)}\ \beta_n^4 = 0 \tag{41}
$$

and

$$
B_i(\beta_k) = \frac{\beta_k^2 M_{nt} + \alpha_i^2\left(\frac{1-\nu}{2I_n}\right)}{\beta_k\alpha_i\left[K_{nt}\nu_n + \left(\frac{1-\nu}{2I_n}\right)\right]} \qquad (i = 1, 2). \tag{42}
$$

The constant $C(\beta_n)$ will be determined from boundary conditions (36).

The system of stresses corresponding to (38) can be obtained by use of relations (23). The resulting expressions can be put in the following form:

$$
\begin{aligned}
t_z^{(o)}(\eta,\rho) &= \sum_{n=1}^{\infty} a_n\ \tau_{11}^n(\rho;\beta_n)e^{-\beta_n\eta} \\
t_r^{(o)}(\eta,\rho) &= \sum_{n=1}^{\infty} a_n\ \tau_{22}^n(\rho,\beta_n)e^{-\beta_n} \\
t_{rz}^{(o)}(\eta,\rho) &= \sum_{n=1}^{\infty} a_n\ \tau_{12}^n(\rho;\beta_n)e^{-\beta_n} \\
t_\theta^{(o)}(\eta,\rho) &= \sum_{n=1}^{\infty} a_n\ \tau_{33}^n(\rho;\beta_n)e^{-\beta_n}
\end{aligned}
\tag{43}
$$

where

$$
\begin{aligned}
\tau_{11}^k &= -\beta_k M_{nt}\ u_1^k + \nu_n K_{nt}\ u_2^{k\bullet} \\
\tau_{22}^k &= K_{nt}\left[(1-\nu)u_1^{k\bullet} - \beta_k\nu_n u_1^k\right] \\
\tau_{12}^k &= \left(\frac{1-\nu}{2I_n}\right)\left(u_1^{k\bullet} - \beta_k u_2^k\right) \\
\tau_{33}^k &= -\beta_k N_{nt}\ u_1^k + \nu_n K_{nt}\ u_2^{k\bullet}.
\end{aligned}
\tag{44}
$$

Application of the homogeneous boundary conditions (36) now yields

$$
\frac{(B_2\alpha_2 - \beta_n\nu)(\alpha_1 - B_1\beta_n)}{\left[(1-\nu(B_1\alpha_1 - \beta_n\nu\right](\alpha_2 - B_2\beta_n)} = \left(\frac{\tanh \alpha_2}{\tanh \alpha_1}\right)^{\pm 1} \tag{45}
$$

and

$$
C(\beta_n) = \frac{(B_2\alpha_2 - \beta_n\nu)\cosh \alpha_2}{\left[(1-\nu)B_1\alpha_1 - \beta_n\nu\right]\cosh \alpha_1} \quad \text{(symmetric)} \tag{46}
$$

$$C(\beta_n) = \frac{(B_2\alpha_2 - \beta_n\nu)\sinh\alpha_2}{\left[(1-\nu)B_1\alpha_1 - \beta_n\nu\right]\sinh\alpha_1} \text{ (Antisymmetric)} \tag{47}$$

On solving (41) for α_1^2, α_2^2 and then substituting these values into equation (45), a characteristic equation for the β_n's is obtained. For an isotropic material (45) reduces to

$$\sin 2\beta \pm 2\beta = 0.$$

We now determine the constants a_n introduced in the series solution (38) from the satisfaction of boundary conditions (30). Since the system of stresses (43) is a statically admissible stress field, the complementary energy principle which was modified for a complex stress field in [6] can be applied,

$$\begin{aligned} \mathrm{Re} \int_{-1}^{+1} \Big\{ & \left[\omega_z^{(2)}(o,\rho) - 0\right] \overline{\delta t_z^{(o)}(o,\rho)} + \\ & + \Big[\Big[\omega_r^{(2)}(o,\rho) - \Big(-\frac{\nu}{1-\nu^2}\rho - \frac{\nu_n}{1-\nu}\Big\{ -\left[V_r^{(o)}(o) + \right. \\ & \left. + V_z^{(1)I}(o)\right]\rho + V_r^{(o)II}(o)\frac{\rho^2}{2}\Big\}\Big)\Big]\Big] \overline{\delta t_{rz}^{(o)}(o,\rho)}\Big\} d\rho = 0. \end{aligned} \tag{48}$$

A simple substitution shows that the a_n's must satisfy the following system of algebraic equations:

$$\sum_{m=1}^{\infty} l_{mn} a_n = b_m, \tag{49}$$

where

$$l_{mn} = \int_{-1}^{+1} \overline{\tau_{1\alpha}^{m}} u_\alpha^n \, d\rho \tag{50}$$

$$(\alpha = 1, 2)$$

$$b_m = \int_{-1}^{+1} \overline{\tau_{12}^{m}} \omega_r^{(2)}(o,\rho) d\rho \tag{51}$$

and $\overline{(\quad)}$ indicates the complex conjugate of $(\quad)$. The expression for $\omega_r^{(2)}(o,\rho)$ to be used in (51) is given by (30), 2. It can be shown by use of the Hermitian theorem introduced in [6] that

$$\int_{-1}^{+1} \left[\tau_{1\alpha}^{m} u_\alpha^n - \tau_{1\alpha}^{n} \overline{u_\alpha^m}\right] d\rho = 0, \tag{52}$$

and hence $\mathcal{L} = |l_{mn}|$ is a hermitian matrix. This hermitian property considerably simplifies the inversion of equation (49).

6. *Conclusion*

The use of thin shell theory to obtain solutions of certain appropriate elasticity problems introduces inaccuracies in the description of stresses and displacements near the edges of the shell. Even thick shell theories, which represent a more accurate approximation, fail to solve the problem of satisfying the boundary conditions exactly. It was Reiss [1] who first showed that this difficulty can be erased by superimposing on the shell (interior) solution a boundary layer solution which corresponds to the problem of the semi-infinite strip.

In this paper the problem of a cylinder under equal and opposite end loads with boundary conditions approximating those obtained in a testing machine was examined. The solution to this problem was obtained by a superposition of two problems: problem I consisted of the loaded cylinder with no end constraints while problem II removed the radial end displacement obtained in problem I. The solution to problem I consisted of the elementary strength of materials solution for the cylinder. The method of the asymptotic integration of the elasticity equations was used to obtain the shell bending and boundary layer (semi-infinite strip) equations needed to yield a first approximation to the solution of problem II. While the solution of the bending equations was straight-forward, a variational method was used to formulate the solution to the boundary layer problem.

The solution of cylinder problem subject to end conditions other than those discussed in this paper can be obtained by making use of the semi-infinite strip formulation presented by Johnson and Little [5].

NOMENCLATURE

a = mid-surface radius of cylinder

c = half-height of cylinder

E, ν = in-plane elastic moduli

E_t, ν_t, G_t = transverse elastic moduli

ϵ_z, ϵ_θ, ϵ_r = axial, circumferential, and normal strain

γ_{rz} = transverse shear strain

h = cylinder thickness

σ_z, σ_θ, σ_r = axial, circumferential, and normal stress

τ_{rz} = transverse shear stress

z, r = axial and radial coordinates

u_z, u_r = axial and normal displacements

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